

Resource Limited Theories and their Extensions

Paul Benioff

Physics Division, Argonne National Laboratory

Argonne, IL 60439

e-mail: pbenioff@anl.gov

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This work is based on the idea that extension of physical and mathematical theories to include the amount of space, time, momentum, and energy resources required to determine properties of systems may influence what is true in physics and mathematics at a foundational level. Background material, on the dependence of region or system sizes on both the resources required to study the regions or systems and the indirectness of the reality status of the systems, suggests that one associate to each amount, r , of resources a domain, D_r , a theory, T_r , and a language, L_r . D_r is limited in that all statements in D_r require at most r resources to verify or refute. T_r is limited in that any theorem of T_r must be provable using at most r resources. Also any theorem of T_r must be true in D_r . L_r is limited in that all expressions in L_r require at most r resources to create, display, and maintain. A partial ordering of the resources is used to describe minimal use of resources, a partial ordering of the T_r , and motion of an observer using resources to acquire knowledge. Reflection principles are used to push the effect of Gödel's incompleteness theorem on consistency up in the partial ordering. It is suggested that a coherent theory of physics and mathematics, or theory of everything, is a common extension of all the T_r .

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I. INTRODUCTION

As is widely recognized, quantum mechanics and its generalizations, such as quantum field theory, is a highly successful theory. So far it has survived every experimental test. Yet in spite of this, nagging problems remain. The problem of measurement is one. Although the use of decoherence to solve the problem [1, 2] helps in that it explains the existence of the pointer basis in measuring apparatuses, questions still remain [3] that are related to whether quantum mechanics is really a theory of open systems only or whether there is a system such as the universe that can be considered to be closed and isolated. This is the approach taken by the Everett Wheeler interpretation [4, 5].

There are other more fundamental questions such as, why space-time is 3+1 dimensional, why there are four fundamental forces with the observed strengths, what the reason is for the observed elementary particle mass spectrum, and why the big bang occurred. Another basic question relates to why quantum mechanics is the correct physical theory. There are papers in the literature that address some of these questions by attempting to show that if things were different then life could not have evolved or some physical catastrophe would happen [6, 7, 8, 9]. However these are all heuristic after-the-fact types of arguments and do not constitute proofs. The possibility of constructing a theory to explain these things, as a "Theory of Everything" or TOE represents a sought after goal of physics [9, 10, 11, 12, 13].

Another very basic problem concerns the relation between physics and mathematics. The view taken by most physicists is that the physical universe and the properties of physical systems exist independent of and a-priori

to an observers use of experiments to construct a theory of the physical universe. In particular it is felt that the properties of physical systems are independent of the basic properties of how an observer acquires knowledge and constructs a physical theory of the universe. This view is expressed by such phrases as "discovering the properties of nature" and regarding physics as "a voyage of discovery".

A similar situation exists in mathematics. Most mathematicians appear to implicitly accept the realist view. Mathematical objects have an independent, a priori existence independent of an observers knowledge of them [14, 15]. Progress in mathematics consists of discovering properties of these objects.

This is perhaps the majority view, but it is not the only view. Other concepts of existence include the formalist approach and various constructive approaches [16, 17, 18, 19]. These approaches will not be used here as they do not seem to take sufficient account of limitations imposed by physics. These include limitations resulting from the physical nature of language [20].

This realist view of physics and mathematics has some problems. This is especially the case for the widely accepted position that physical systems exist in and determine properties of a space-time framework. However, mathematical objects exist outside of space-time and have nothing to do with space-time. If this is the case, then why should mathematics be relevant or useful at all to physics? It is obvious that they are closely entwined as shown by extensive use of mathematics in theoretical physics, yet it is not clear how the two are related at a foundational level.

This problem has been well known for a long time. It was expressed by Wigner [21] in a paper entitled *The*

Unreasonable Effectiveness of Mathematics in the Natural Sciences. A related question is, Why is Physics so Comprehensible? [22].

Another foundational issue is based on the universal applicability of quantum mechanics. It follows that all systems, including experimental equipment, computers, and intelligent systems are quantum systems in different states. The macroscopic aspect of these systems does not change their quantum mechanical nature.

It follows that the process of validation (or refutation) of any theory, including quantum mechanics, is a quantum dynamical process described by quantum dynamical evolution laws. One sees then that quantum mechanics must in some sense describe its own validation by quantum systems. However almost nothing is known so far about the details of such a description.

These concerns form the background for this paper. This work begins with the observation that there is an aspect of physics that is faced daily by physicists, but is not included in physical or mathematical theories. This is the amount of physical resources, as space, time, momentum, and energy resources, required to carry out experiments and theoretical calculations. For experiments using large pieces of equipment and calculations requiring massive amounts of computing power, the resource requirements can be considerable.

This use of resources is not discussed in a theoretical context because of a strong belief that the amount of resources needed to carry out experiments and make theoretical calculations on different types of systems has nothing to do with the contents of physical theories being created and verified by this process. The material facts of what is true physically and properties of the theories making predictions supported by experiment, are believed to have nothing to do with the space time and energy momentum resources needed to do the experiments and carry out the computations. Extending this belief to a TOE would mean that resource use by the knowledge acquisition process, whose goal is the construction of a coherent theory of mathematics and physics or TOE, has nothing to do with the contents of the TOE.

The main purpose here is to take steps towards the possibility that this may not be correct, especially for foundational properties of physics and mathematics. Included are questions regarding the strengths and existence of the four basic forces, why space-time is 3+1 dimensional, the nature and reasons for the big bang and other general cosmological aspects, and why quantum mechanics is the correct physical theory.

It should be strongly emphasized that the generally believed view of the independence between resource related aspects of carrying out experiments and calculations and the content of the theories created is true for the vast majority of physics and mathematics. There is ample evidence to support this view. Probably the best evidence is that if it were not true, the dependence would have been discovered by now.

However the fact that it is true for most systems and

properties does not mean it is necessarily true for all. In particular, resource related aspects of doing experiments and calculations to create valid physical theories may influence the contents of the theories, at least at a very basic level.

This work takes some initial steps to see if this possibility has merit. The approach taken is an extension of the general ideas presented in [25] and [20] and in references cited therein. The idea is to describe resource limited domains, theories and languages, Each theory and domain is based on a limited amount of physical resources available to verify or refute the statements in the language. The relative strength of each theory depends on the amount of available resources. Theories with more available resources are stronger than those with less.

The next two sections give informal arguments that give some support to the possibility suggested here, that resource use may influence the basic contents of physical theories. The arguments are based on the relation between resource requirements and the size of the region or system being investigated. Another relation discussed is that between the indirectness of the reality status of systems and their size [25].

These arguments lead to a description of resource limited theories, languages, and domains. This is provided in the subsections of Section IV. Included are a brief description of physical resources and a description of procedures, instructions, equipment, and purposes of equipment and procedures as components of the theory domains and languages. Other components include symbols strings as outputs of measurements and computations, and the implementation operation. These components are used to give descriptions of agreement between theory and experiment, and of theorem proofs in the theories (subsections IV C 5 and IV C 6). Also the minimum resources required to determine the truth value of statements about properties of systems are discussed. The final subsection gives details on the effect of resource limitations on language expressions.

Section V describes the use of the partial ordering of the physical resources to partially order the resource limited theories. The following section describes briefly the dynamics of an observer using resources to acquire knowledge and develop physical and mathematical theories. The relation to the theories in the partial ordering is also noted.

A characteristic of resource limited theories is that each theory includes parts of arithmetic and other theories. As such one expects Gödel's incompleteness theorems [23, 24] to apply. It is assumed that the resource limitations do not affect the validity of these theorems. One concludes from the second theorem that none of the theories can prove their own consistency, and that the same incompleteness applies to any extension proving the consistency of the first theory.

It is possible to iterate the extension process and push the effect of Gödel's theorem from theories with less available resources to theories with more available resources.

This is discussed in Section VII by the use of reflection principles [26, 27] that are based on validity. Because of the resource limitations the reflection principles have to be applied separately to each individual sentence rather than to all sentences at once in a theory.

Limit and consistency aspects of a TOE are discussed in Section VIII. The possibility that a coherent theory of physics and mathematics, or a TOE is a common extension of all the theories is noted as is a problem that consistency poses for a TOE. The final section summarizes the paper and points out the need for work on aspects not considered here.

It must be emphasized that the goal of this paper is to describe some properties of resource limited theories, domains, and languages, and the motion of observers using resources to develop theories. As such this work is only a very small initial step in the approach to a coherent theory of physics and mathematics or a TOE. The material presented here *does not represent in any way* a completed TOE capable of verification or refutation. To achieve this many important aspects, not treated here, must be described. These include but are not limited to probability and information theory aspects, a much more detailed description of available physical resources including a description of resources *within each theory*, and specification of the axioms of the theories. Also many of the well known physical theories, such as quantum mechanics, general relativity, and possibly string theory, would have to be included in some form.

II. RESOURCES AND REGION SIZE

It is useful to begin by noting the relation between theories and the size of the systems and regions to which the theories apply. For regions whose size is of the order of the Planck length, $\sim 10^{-33}$ cm, string theory is used. For Fermi sized regions, $\sim 10^{-13}$ cm, the strong interaction is dominant with QCD the appropriate theory. For larger regions, $\sim 10^{-8}$ cm up to thousands of cm in size, electromagnetic interactions are dominant with QED the appropriate theory. Finally for cosmological sized regions, up to 10^{28} cm in size, gravity is the dominant interaction with general relativity the appropriate theory.

It is also well known that to investigate events in a region of size r , probes with momentum $\geq \hbar/r$ and energy $\geq \hbar c/r$ must be used. The latter follows from the fact that the characteristic time associated with a region of size r is given by the time, r/c , it takes light to cross the region. Here \hbar is Planck's constant divided by 2π and c is the velocity of light. This sets a lower limit on the energy momentum of a probe required to investigate events in regions of size r . It is a significant restriction for small r .

What is, perhaps, not appreciated, but is well known by both theoretical and experimental physicists, is the fact that there is another scale of physical resources associated with these regions of different sizes and their

associated theories. These are the space time and energy momentum resources needed to carry out theoretical calculations and do experiments for the theories and their systems relevant to regions of size r .

The relationship between the size of the region investigated and the resources needed can be set out in general terms for both experiments and theory based computations. At present it appears impossible to do meaningful experiments and calculate the associated predicted outcomes for Planck sized objects as one does not know what to do or even if such objects exist. Because these objects are so small an extremely large or even infinite amount of resources are needed for such experiments and computations.

To investigate Fermi sized objects, large accelerators and large amounts of energy are needed to produce the particle beams and maintain the relevant magnetic fields. Computations are resource intensive because the strong interaction makes a perturbation approach to QCD computations infeasible. The resources needed are large, but finite. Less resources are needed for relevant calculations and experiments on atomic and larger systems. However more resources, in terms of very large telescopes, on and near earth, and long viewing times with very sensitive detectors, are needed to investigate cosmological sized objects, especially those that are very far away.

The relations between resources needed and the size of the region investigated is shown schematically in Figure 1. The ordinate shows a characteristic size parameter of the object being investigated. The upper limit shows the present age of the universe in cm and the lower limit is the Planck length in cm.

The first abscissa label, resource use, denotes the amount of resources required to carry out theoretical predictions and to do experiments on the object being investigated. The amounts increase from left to right as shown by the arrow. Values are not given because it is at present an open question how to quantify the resources required. Also for this paper there is no need to quantify the resources.

The curve is a freehand or arbitrary schematic representation of the dependence of the resources required to carry out experiments and theoretical calculations on the characteristic size of the object being investigated. The arbitrariness, or lack of knowledge of curve details, is denoted by the dashes in the curve. In spite of this a curve, such as that in Fig. 1, is useful to represent some properties of the dependence that one does know. This is that the curve has two branches and that each branch must approach a limit. Here these limits are taken to be the Planck length and c times the age of the universe. If one feels these limits are too restrictive they may be changed. The important point is that there seem to be such limits.

The presence and location of the minimum represents systems whose size, real or perceived, is such that we can directly observe them. Many of these objects can be directly examined and handled to determine directly

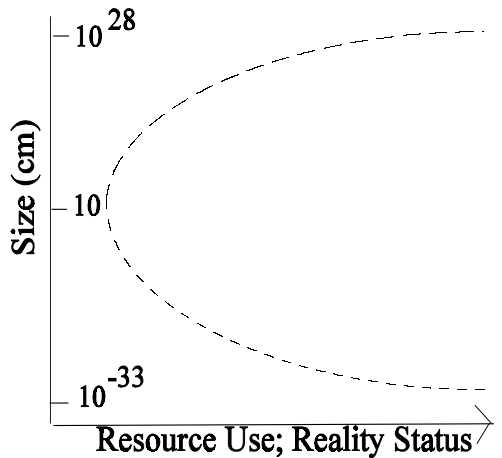


FIG. 1: A schematic plot of the resource use and indirectness of reality status for systems of different sizes. Resource use refers to the amount of resources needed to carry out calculations and experiments. Reality status is a measure of the number of layers of theory and experiment needed to give properties of systems. Additional details are in the text.

observable properties. No experiments or theory is required as the properties can be determined directly by our senses. Included are such properties as "this rock is heavy, hard, and brown", "the horizon looks flat", "the sun is hot, bright, and moves through the sky". The size of the sun is not the actual size but is the size perceived by us, which is a few cm.

These directly perceived properties belong in the region of minimal resources required because they are direct and uninterpreted. No theory or experiment is used to explain why anything happens or what its physical properties are. The resource location of the minimum of the curve is arbitrary. It is not set at 0 resources to allow freedom in the choice of how resources are quantified.

The ordinate location of the minimum represents sizes of objects that can be directly observed or perceived. It is a broad minimum ranging over sizes of the order of 1 cm to 100 cm. To reflect this the minimum is arbitrarily set at 10 cm. The size range of the minimum is also representative of our size. The reason is that our size is of the order of the (real or perceived) sizes of all systems that we can directly experience.

III. SIZE AND INDIRECTNESS OF THE REALITY STATUS

There is another quite different aspect of theories, theoretical calculations, and supporting or refuting experiments that is relevant to Figure 1. This is the indirectness of the reality status as a function of the size of physical

systems [25].

To see this one notes that the validity of an experimental test of a theoretical prediction depends on the fact that each piece of equipment used in the experiment is properly functioning. But the proper functioning of each piece of equipment depends in turn on other supporting theory and experiments which in turn \dots . As an example suppose an experiment to test the validity of a theory at some point uses two pieces of equipment, E_1, E_2 . The validity of this experiment as a test depends on the proper functioning of E_1 and E_2 . However, the proper functioning of E_1 also depends on some theory which may or may not be the same as the one being tested, and also on some other experiments each of which depend on other pieces of equipment for their validity. This argument then applies also to the experiments used to validate the theory on which the proper functioning of E_1 is based. Similar statements can be made for the proper functioning of E_2 .

Basic examples of such equipment are those that measure time and distance. The truth of the assertion that a specific system, called a clock, measures time depends on the theory and experiments needed to describe the functions of the clock components and the proper functioning of the clock components. The conclusion that a particular piece of equipment measures time depends on the conclusions that each component of the equipment functions properly. Similar arguments can be made for distance measuring equipment and equipment for measuring other physical parameters.

Computations made to generate theoretical predictions have the same property. A computation is a sequence of different steps each performed by one or more pieces of equipment such as a computer. Here the proper functioning of the computer depends on theory, which may or may not be the same as the one for which the computation is made, and on experiments that support the theory needed to assert that the computer does what it is supposed to do.

These arguments show that the validity of an experiment or theoretical computation depends on a downward descending network of theories, computations, and experiments. The descent terminates at the level of the direct, elementary observations that were discussed before. As was noted these require no theory or experiment as they are uninterpreted.

The indirectness of the reality status of systems and their properties is measured by the depth of descent between the property statement of interest and the direct elementary, uninterpreted observations of an observer. This can be described approximately as the number of layers of theory and experiment between the statement of interest and elementary observations. The dependence on size arises because the descent depth, or number of intervening layers, is larger for very small and very large systems than it is for moderate sized systems.

This line of argument gives additional support to the basic nature of the direct elementary observations perceived by an intelligent system. It is also shown by the

curve in Figure 1 with the second abscissa label as a measure of the indirectness of the reality status of different sized objects. The indirectness can be roughly represented by the number of layers of theory and experiment between elementary observations and the theory calculations and experiments that are relevant for the object being investigated.

The relation between the two abscissas suggests that resource use can be included by considering resource limited theories, domains, and languages and their relation to observers use of resources to develop theories. Initial steps in this direction are carried out in the following sections.

IV. RESOURCE LIMITED THEORIES, DOMAINS AND LANGUAGES

Before describing resource limited theories, domains, and languages, it is useful to give a brief description of physical resources.

A. Physical Resources

Here physical resources are considered to consist of space, time, momentum, and energy. If space and time is $d+1$ dimensional, then the amount, r , of resources available is a $2d+2$ dimensional parameter $r_1, r_2, \dots, r_{2d+2}$. Each of the parameters can be taken to be continuously varying or it can be considered to be discrete.[37] Since the concerns of this paper are independent of which choice is made, the choice of a discrete or continuous r will be left to future work.

Each parameter, r_j , of the $2d+2$ parameter description of r is a number indicating the amount of the j th resource available. The d space parameters $r_{[1,d]} = \{r_1, \dots, r_d\}$ and one time parameter r_{d+1} give the amount of space and time available. Similarly the d momentum parameters r_{d+2}, \dots, r_{2d+1} and energy parameter r_{2d+2} give the amount of momentum and energy available.

Here it is also useful to consider a resource space whose elements are the $2d+2$ dimensional r . The space has a partial ordering given by that defined for the resources. That is $r \geq r'$ if $r_j \geq r'_j$ for all $j = 1, \dots, 2d+2$. This space represents a background for description of the resource limited theories and motion of observers developing theories.

This description is sufficient for this paper even though it is quite superficial and brief. Additional details, including quantification and other aspects, are left to future work.

B. Basic Resource Limitations

Let T_r, D_r, L_r be a theory, domain, and language associated with each value of r . L_r is the language used by

T_r and D_r is the domain or universe of discourse for T_r . Here r is the maximum amount of space, time, momentum, and energy resources available to T_r, L_r, D_r . This puts limitations on the T_r, L_r, D_r .

A domain D_r is limited by the requirement that at most r resources are needed to determine the truth value of any statement S in D_r . Let $r(S)$ be the resources needed to determine the truth value of S , i.e. to verify or refute S . If S is in the domain D_r , then

$$r(S) \leq r. \quad (1)$$

If more than r resources are needed to verify or refute S , then S is not in D_r .

The statements S can be quite general. Included are statements about properties of procedures, instructions, equipment, computers, and many other physical and mathematical objects. Since S often includes statements about procedures used to determine properties or systems, there can be many statements S for a given system and property, each based on a different procedure and with a different value of $r(S)$. Similarly properties can be quite general. Included are properties related to experimental tests of theories, purposes of procedures and instructions, existence of systems, etc.. The main point is that D_r is limited to those S that satisfy Eq. 1.

Here the value of $r(S)$ is considered relative to the basic uninterpreted directly perceived properties. Any resource value associated with these properties is the zero point. Thus for each S $r(S)$ includes the resources needed to construct all the equipment needed to verify or refute S .

Note that D_r is closed under negation as $r(S) = r(\neg S)$ (\neg means not). However D_r is not closed under conjunction or disjunction. For conjunctions this follows from

$$r(S), r(T) \leq r(S \wedge T) \leq r(S) + r(T). \quad (2)$$

One sees from this that it is possible that S and T are such that $r(S) \leq r$ and $r(T) \leq r$ but $r(S \wedge T) > r$. In this case S and T are in D_r but $S \wedge T$ is not. Note too that $r(S \wedge T) < r(S) + r(T)$ occurs if procedures for determining the truth values of S and T use some of the same equipment. Also $r(S \wedge T)$ should be such as to avoid double counting of resources used to construct equipment used in both procedures. The same arguments hold for disjunctions as

$$r(S), r(T) \leq r(S \vee T) \leq r(S) + r(T). \quad (3)$$

The theories T_r are limited by the requirement that proofs of all theorems of T_r require at most r resources to implement. Thus S is a theorem of T_r if a proof of S can be done using at most r resources. If S requires more than r resources to prove, then S is not a theorem of T_r .

This limitation follows directly from the physical nature of language [20]. If the physical representation of expressions of L_r corresponds to states of systems in D_r ,

which is the case assumed here, then the representation corresponds to a Gödel map of the expressions into system states in D_r . In this case the provability of a statement corresponds to a statement about properties of systems that are in D_r . As such, the proof statements are subject to the limitations of Eq. 1.

Another limitation on T_r is that (assuming consistency) all theorems of T_r must be true in D_r . It follows from this and the first limitation that no statement can be a theorem of a consistent T_r if it is false in D_r , requires more than r resources to verify, or more than r resources to prove.

The language L_r must satisfy a limitation based on the physical nature of language. All expressions X in L_r as strings of symbols are limited by the requirement that they need at most r resources to create, display, and maintain. This includes symbol strings, as strings of numerical digits (i.e. as names of numbers), which are used in all computations, quantum or classical, as outputs of measurements, and as instructions or programs for experimental or computation procedures. It is possible that there are expressions in L_r which are sentences but have no interpretation as statements in D_r because the interpretation does not satisfy Eq. 1.

In this paper some major simplifying assumptions are made. One is that there is no discussion about how the resources and the limitations are described within the statements of T_r . All resource discussions here are assumed to take place in the metatheory of the theories T_r . This puts off to future work removal of this assumption, which is clearly necessary.

Another assumption is that probabilistic and information theoretic aspects are not included here. It is clear that this assumption must be removed if quantum mechanics is to be included in any detail. This is especially the case if the universal applicability of quantum mechanics is taken into account.

A third assumption is that one specific physical representation of the symbols and expressions of L_r is assumed. Specific details are not given here as an abstract representation is sufficient.[38] It is clear, though, that there are many different physical representations of expressions, each with their own resource characteristics.

C. Contents of the Theories and Domains

1. Procedures, Instructions, Equipment

Included in the domains of the theories are processes or procedures, instruction strings, equipment, and statements about the function or purposes of procedures or equipment, and other physical and mathematical systems. Associated with a process or procedure P is a set of instructions I_P (as a symbol string) for using several pieces of equipment. Here $E_P = \{E_1, \dots, E_n\}$ denotes the equipment used by P . I_P may also include instructions for assembling the equipment in E_P in specified

locations and instructions on when to use it. In this case E_P includes equipment to measure space and time.

Procedures also contain branches. An example is the procedure P : “Use E_3 to place E_2 3 meters away from E_1 . Activate E_1 and E_2 . Read outcome of using E_2 , if outcome is 01101 do P_1 if outcome is 11010 do P_2 ”. Here P_1 and P_2 are two other procedures that may or may not contain branches.

There are no specific limits placed on pieces of equipment E . E can be as simple as clocks and measuring rods or as complex and massive and large as telescopes and particle accelerators. Of course, larger more complex equipment requires more resources to assemble, use, and maintain than does smaller, less complex equipment.

It is important here to clearly separate *purposes* of both procedures P and equipment E from *use* of P and E . I_P should not say anything about what P does or what any equipment used in P does or why it is used. No theory is involved or needed to carry out I_P . I_P represents instructions that can be followed by robots, automata, or other well trained implementers. Implementers, such as robots, must be able to follow instructions very well without knowing what anything is for.

The example of a branching P given above, violates this requirement by saying what E_3 does, “Place E_2 3 meters away from E_1 ”. This was done both for illustrative purposes and as an aid to the reader. A proper description of I_P would include instructions for how to use E_3 without saying anything about what E_3 is used for (space measurement). A possible way of saying this might be “activate E_3 , move E_1 until outcome 3 shows on E_3 ”.

The same holds for the activation part of P . This denotes a procedure such as plugging cords into an electric socket. The implementer need not know that the procedure turns on E_1 and E_2 in order to follow the instructions. Activation may include observation of lights to determine if the equipment is on and properly functioning.

The example P also includes the component “Read outcome on E_2 , if outcome is 01101 do \dots ”. This implies the direct reading of a symbol string showing in some part of E_2 . No equipment is used as this is a direct uninterpreted observation. No theory is used to make the observation and the implementer does not have to know whether the outcome is or is not a number or a symbol string to compare it with 01101.[39] However the procedure may include instructions that are equivalent to using a piece of equipment E_4 to read E_2 . This is useful in case it is difficult to read the output of E_2 and it is much easier to read the output of E_4 than of E_2 .

2. Purposes

Associated with each procedure P , equipment E , and instruction string I , is a purpose A . These denote what the procedure, piece of equipment, or instruction string

does. Examples of A for procedures are “prepares a system in state ρ to n figures”, “measures observable O to n figures”, “computes $Tr\rho O$ to n figures”, “measures time to n figures”. For equipment, examples are “is a telescope with operating parameters —”, and “is an accelerator with operating parameters —”, and for instructions, examples are “is instructions for using P ”, etc.. The reason for the accuracy phrase “to n figures” will be discussed later.

The empty purpose, “has no purpose”, is also included. This accounts for the fact that most processes do nothing meaningful, and most states of physical systems are not pieces of equipment that do anything meaningful. Also most symbol strings are not instruction strings or are instruction strings for meaningless procedures. For example making a pile of rocks in a road may have a purpose as a barricade but this is not relevant here.

Purpose statements are used to associate purposes with procedures, equipment and instruction strings. The statement $F(P, A)$ means that “ A is the purpose of P ”. If A is “measures time to n figures”, then $F(P, A)$ is the statement “ P measures time to n figures”. Depending on what P and A are $F(P, A)$ may be true or false. In a similar fashion $F(E, A)$ and $F(I, A)$ are purpose statements for E and I .

Another type of useful purpose statement refers to the truth value of a statement S . If B is the purpose statement “determines the truth value of S ”, then $F(Q, B)$ means “ Q determines the truth value of B ”. Note that S can be a statement $F(P, A)$. Then $F(Q, B)$ means “ Q determines the truth value of $F(P, A)$ ”.

This raises the question about the possibility of an infinite regress where $F(Q_{n+1}, B_{n+1})$ says that Q_{n+1} determines the truth value of B_{n+1} and B_{n+1} is the purpose “determines the truth value of $F(Q_n, B_n)$ ”. There is indeed a regress but the regress is finite. The reason is that the procedures, equipment and theory become more elementary as n increases with the regress terminating at the level of direct uninterpreted sense impressions of the type discussed in Section II. The regress corresponds to a descent through a network of theories, computations, and experiments, Section III.

This can be seen directly by noting that use of a procedure P in some experiment requires that the purpose statement $F(P, A)$ be true. Clearly any procedure Q whose intended purpose is to determine if $F(P, A)$ is true or false must be more basic and direct, and depend on theories and experiments requiring less resources and interpretation than that for the experiment using P . It is clear that this avoids circular situations where the validity of the purpose statement, $F(Q, B)$, with the purpose B given by “determines the truth value of $F(P, A)$ ”, depends on a theory whose validity is being tested by an experiment using P .

3. Outputs as Symbol Strings

As the above shows, outputs as finite strings of symbols are an essential part of procedures. Any measurement or calibration equipment used in a procedure generates output. It is also worth noting that any output that is a string of n digits, does not in general denote a number. Instead it is an n figure representation of a number.

It is worthwhile to discuss this a bit especially in view of the resource limitations on the T_r . The 4 digit output binary string 1000 corresponds to a natural number as it is a name for one. However output in the binary form of 1×10^{11} does not correspond to a natural number. Instead it is a one figure representation of some range of numbers. However 1.000×10^{11} is a natural number (binary base and exponent) as it is equivalent to 1000.

The situation is similar for output strings considered as rational numbers. For instance the 6 digit binary output 101.011, which is equivalent to 101011×10^{-11} , does not correspond to a specific rational number. Rather it corresponds to a 6 figure representation of some range of rational numbers. The point is that if one assumes that an output string such as 101.011 of some measurement is a rational number, then one is led to the conclusion that $101.011 + \epsilon$, where ϵ is an arbitrarily small rational number, is not the output of the measurement. While this is literally true it can quite easily lead to wrong conclusions about the accuracy of the measurement, namely that the measurement is infinitely accurate. Similar arguments hold for real numbers in that no output digit string represents a real number[40]

This description for the binary basis extends to any k – ary basis with $k \geq 2$. However, the possible values of k are limited because there is a limit in how much information can be packed into a given space-time volume [28].

The same limitations hold for purposes A of procedures P . If P requires at most r resources to carry out and P represents a measurement of a continuously varying property, such as momentum, then the purpose statement $F(A, P)$ must include the property measured and the number of figures used to represent the outcome. If P measures momentum, or prepares a system in some quantum state ψ , then $F(A, P)$ must say “ P measures momentum to n figures” or “ P prepares state ψ to n figures”. A procedure P that measures momentum or prepares ψ with no n figure qualifier, would require an infinite amount of resources to implement. Also the outputs of some of the equipment used in P , would have to be real numbers and require an infinite amount of resources to display.

For measurements of discrete valued properties such as spin projections in quantum mechanics, the “ n figure” qualifier can be dropped. However this is the case only if P does not also measure the continuously variable direction of the magnetic field serving as the axis of quantization.

4. Implementation

As described the procedures P and their associated instructions I_P do not include their own implementation. Also most P and I_P do not include instructions on when and where they are to be implemented.

This is taken care of by use of an implementation operation Im . This operation refers to the actual carrying out of a procedure P by use of the instructions I_P . Implementation of P also needs to specify when and where P is to be done. This is done by use of procedures P_{s-t} that measure space and time to n figures. The value of n depends on the procedure used.

Im operates on pairs of procedures P, P_{s-t} and on $d+1$ tuples \underline{x} of n figure binary strings. The result of actually implementing P at a location and time given by \underline{x} , as determined by use of P_{s-t} , is denoted by $Im(P, P_{s-t}, \underline{x})$. Since P uses equipment, I_P must describe how to set up the equipment and how to use it to implement P . $Im(P, P_{s-t}, \underline{x})$ then puts the equipment used in P in some final state.

Many procedures are measurements or computations. In this case the outcome as a string of digits corresponds to part of the final state of the equipment used. Define Ou to be the operation that picks out the output. In this case $Ou(Im(P, P_{s-t}, \underline{x}))$ is the outcome digit string obtained by implementing P at \underline{x} as determined by P_{s-t} .

The implementation operation is quite separate from procedures P and their instructions I_P . This is the case even for I_P that state when and where P is to be carried out. Also I_P often include instructions regarding relative spacing and delay timing of the various components. In this sense the I_P are similar to construction and operating manuals accompanying disassembled equipment. Operating manuals can talk in great detail about using equipment or implementing procedures, but this is quite different from the actual use or implementation.

It should be noted that physical resources must be used to carry out the implementation operation. For any procedure P resources are considered to be used at the space time point at which Im is carried out. This includes the location of the space and time region needed to implement P and the momentum and energy resources used to implement P in the space time region so located.

To see how this works let P be a procedure whose purpose is denoted by A and $E_P = E_1 \cdots, E_n$ be the equipment used by P . The truth of $F(P, A)$ namely, that P does what it is supposed to do, depends on the truth of $F(E_P, A_P) = \wedge_{j=1}^n F(E_j, A_j)$ where A_j is the purpose of E_j .

Let $Im(P, P_{s-t}, x)$ be the result of implementing P at x by use of P_{s-t} . One requires at a minimum that $F(E_P, A_P)$ be true over the space and time region associated with the point x at which P is implemented. This would be especially relevant for procedures whose implementation destroys some of the equipment used.

Let Q be a procedure whose purpose is to verify or refute $F(E_P, A_P)$. That is, Q is a procedure to ensure

that all the equipment in E_P is properly working. Implementation of Q at x' gives outcome $1(0)$ if $F(E_P, A_P)$ is true (false) at x' , or

$$Ou(Im(Q, P_{s-t}, x')) = 1 \implies F(E_P, A_P). \quad (4)$$

Since $x' \neq x$ in general, physical theory (and equipment monitoring as part of P) is used to ensure the truth of $F(E_P, A_P)$ for the space time region occupied by the implementation of P at x . The theory used includes basic aspects such as the homogeneity and isotropy of space and time, and predictions regarding a small influence of the environment on the equipment in E_P in going from x' to x .

5. Agreement between Theory and Experiment

The contents of the T_r and D_r described so far can be used to describe procedures that are tests of agreement between theory and experiment. Here only a very simple situation is considered in which one single experiment and one single theoretical computation is sufficient to test for agreement between theory and experiment. Discussions of tests that require use of statistics and repeated experiments will be deferred to future work when probability concepts are introduced.

The instructions I_P include instructions for the use of three procedures. Included are P_{ex} , whose purpose is to measure a property specified to n figures on a system prepared in a state specified to n figures, P_{s-t} to measure space and time to n figures, and P_{th} to compute a number to n figures. The measurement will also give an n figure result. For simplicity the same value of n is used for each procedure.

The output symbol string, computed by P_{th} , is an n figure representation of a numerical theoretical prediction for the experiment. As such it represents a theorem of the theory being tested where the theorem is adjusted to take account of the n figure specifications of the system state and property being measured and the output of the measurement.

Let A_{ex}, A_{s-t}, A_{th} denote n figure purpose phrases for P_{ex}, P_{s-t}, P_{th} . A_{ex} says "measures to n figures a property Q specified to n figures on a system in a state α specified to n figures". A_{s-t} says "measures space and time to n figures", and A_{th} says "computes to n figures the theoretical value for the n figure specification of property Q measured on a system in the state α specified to n figures".

The statement of agreement between theory and experiment for these procedures is the statement

$$Ag \equiv Ou(Im(P_{ex}, P_{s-t}, x_{ex})) = Ou(Im(P_{th}, P_{s-t}, x_{th})). \quad (5)$$

Ag says that the outcome of implementing P_{ex} at x_{ex} determined by use of P_{s-t} equals the outcome of implementing P_{th} at x_{th} determined by use of P_{s-t} .

The goal is to determine the truth value of Ag . The truth of Ag is a necessary, but not sufficient, condition for agreement between theory and experiment for the prediction that system in state α has property Q . The other necessary condition is that the three procedures have the purposes A_{ex}, A_{th}, A_{s-t} . This is expressed by the requirement that the statement

$$Pur \equiv F(P_{ex}, A_{ex}) \wedge F(P_{s-t}, A_{s-t}) \wedge F(P_{th}, A_{th}) \quad (6)$$

must also be true. The truth of both Ag and Pur is necessary and sufficient for agreement between theory and experiment at α, Q .

The usual way of testing for agreement between theory and experiment is to actually implement the procedures as described here to determine if Ag is true or false. This assumes the truth of Pur , which is based on other experiments and theory that agrees with experiment at other points.

The well known use of resources to carry out experiments and theoretical computations is seen here by the requirement that resources are needed to verify or refute both Ag and Pur . If $r(Ag)$ and $r(Pur)$ denote the resources needed, then Ag and Pur are in D_r and T_r if $r > r(Ag)$ and $r > r(Pur)$. The notion that Pur and Ag might also be theorems of some T_r , with resulting additional resource needs, is an intriguing but unexplored possibility.

6. Proofs of Theorems in T_r

The contents of the T_r can also be used to describe proofs of sentences in L_r . To see how this works, let S be some statement such that S is a theorem of T_r , or

$$T_r \vdash S. \quad (7)$$

This means that there exists a proof, X , of S in T_r where X is a string of formulas in L_r such that each formula in X is either an axiom of T_r or is obtained from some formula already in X by use of a logical rule of deduction.

With no resource limitations, which is the case usually considered, the process of determining if T_r proves S consists of an enumeration X of theorems of T_r . If S is a theorem it will appear in X after a finite number of steps. The proof X with S as a terminal formula will have a finite length. If S is not a theorem it will never occur in an X and the process will never stop.

Eq. 7 is a statement in the metalanguage of the theories T_r . To give a corresponding statement in L_r use is made of the physical representation of expressions in L_r . It was noted in subsection IV B that if a physical representation of the expressions of L_r is in D_r , then it corresponds to a Gödel map G of the expressions into states of systems in D_r .

In this case theoremhood can be expressed using the contents of the T_r , Section IV C. Let P be a procedure acting on the states of physical systems described above.

Let α be a state of some of the systems and A_α a purpose phrase in D_r that says in effect “repeatedly generate different states of the systems by a (specified) rule. If and when state α appears on the designated subsystems, stop and output 1”.

Let B_S , be a purpose phrase in the metalanguage that says “enumerates proofs based on the axioms Ax_r and stops with output 1 whenever S is produced at the end of a proof”. Now require that $\alpha = G(S)$ and that A_α satisfies

$$G(B_S) = A_{G(S)}. \quad (8)$$

This requires A_α to be a physical purpose phrase that is equivalent under G to the purpose phrase for a proof enumeration until S is generated.

The statement that P is a proof of S of T_r is given by the sentence Y

$$Y \equiv F(P, A_{G(S)}) \wedge F(P_{s-t}, A_{s-t}) \wedge Ou(Imp(P, P_{s-t}, x)) = 1. \quad (9)$$

Here $Ou(Imp(P, P_{s-t}, x)) = 1$ says that the output of implementing P at x , based on use of P_{s-t} , is 1. This means the procedure stopped and P is a proof of S under G . The sentences $F(P, A_{G(S)})$ and $F(P_{s-t}, A_{s-t})$ are statements about the purposes of P and P_{s-t} .

Theoremhood for S in T_r is expressed by a sentence $Th_r(S)$ in L_r saying that for all x there exist procedures P, P_{s-t} that satisfy $Y \equiv Y(P, P_{s-t}, G(S), x)$:

$$Th_r(G(S)) \equiv \forall x \exists P, P_{s-t} Y(P, P_{s-t}, G(S), x). \quad (10)$$

If there is no such procedure then S is not a theorem of T_r . Note that because the T_r are incomplete, it does not follow from S not being a theorem that the negation of S is a theorem. Each sentence is a theorem of T_r if and only if it can be proved with a procedure requiring less than r resources to implement.

Axioms play an important role in theories as they represent the input sentences for proofs. At this point it is not possible to specify the axioms, Ax_r , for each T_r . However some aspects are known. All Ax_r consist of two components, the logical axioms and the nonlogical axioms. The logical axioms and logical rules of deduction are common to all theories as they represent a formal codification of the rules of thought and logical deduction used to develop theories and to acquire knowledge. The nonlogical axioms distinguish the different theories as they should express exactly what a theory is about.

Also all Ax_r are limited by the requirement that each sentence in Ax_r as a theorem of T_r must satisfy the resource limitations on theorems of T_r stated earlier. This has the consequence that for very small values of r the T_r are quite fragmentary as they contain very few sentences and even fewer as theorems. The resource limitations become less restrictive as r becomes large.

Subject to the above limitations all the Ax_r would be expected to include axioms for arithmetic and axioms

for operations on binary (or higher) names of numbers as $\bar{0}$ – $\bar{1}$ symbol strings. This includes the use of these strings in expressions in L_r corresponding to informal subscript and superscript labelling of variables, constants, functions and relations. Unary names are not used because arithmetic operations on these are not efficiently implementable [34].

The string axioms needed are those defining a concatenation operator, $*$, projection operators on different string elements, and string symbol change operators. Also included are two functions from strings to numbers denoting the length of a string and the number value of a string.

It is expected that the Ax_r will also include axioms for quantum mechanics and other physical theories. Further specification at this point is neither possible nor useful. The reason is that axioms and logical rules of deduction are in essence the initial conditions and dynamical rules for theorems of theories. As such one wants to first investigate the theories in more detail to see what properties they should have. This includes study of the dynamics of observers using resources to develop valid theories and inclusion of probabilistic and information theory aspects. Study of these and other aspects would be expected to give details on the specification of the Ax_r .

D. Minimal Use of Resources

It is of interest to see in more detail how the basic resource limitations of subsection IVB apply to the T_r . The main use of resources occurs through the implementation operation. This occurs because for any statement S the resources needed to verify or refute any statement S are used by implementing the various procedures appropriate to S . This applies to all statements, including purpose statements, such as $F(P, A)$, provability statements, existence statements for different types of physical systems, and all others.

A well known aspect of physics and other theories is that there are many different ways to prove something or to experimentally test some property of systems or to do things in general. This is expressed here by procedure specific sentences such as those of Eqs. 5, 6, and 9.

Let $S(\underline{P})$ be a procedure specific statement asserting that use of the procedures \underline{P} shows that a specified system has a specified property. The underlined \underline{P} denotes possible use of more than one procedure. This is seen in the Im operation that operates on 2 procedures and the Pur and Ag statements based on 3 procedures.

Let $r(S, \underline{P})$ denote the resources needed to verify or refute $S(\underline{P})$. Since $r(S, \underline{P})$ is procedure dependent, there must be a set of procedures \underline{P}_{min} that minimizes $r(S, \underline{P})$. In this case

$$r(S, \underline{P}_{min}) = \min_{\underline{P}} r(S, \underline{P}) \quad (11)$$

is the least amount of resources needed to verify or refute a procedure specific statement $S(\underline{P})$.

Let S be the procedure independent statement asserting that a specified system has a specified property. Then $r(S, \underline{P}_{min})$ is also the least amount of resources needed to verify or refute S . Define $r(S)$ by

$$r(S) = r(S, \underline{P}_{min}). \quad (12)$$

Here $r(S)$ is the least amount of resources needed to verify or refute S .

Note that one does not verify or refute S by hunting through all possible procedures. Instead one sets up procedures based on accumulated knowledge and resources spent. After a few tries one either succeeds in which case a procedure (or procedures) satisfying some $S(\underline{P})$ has been found. In this case the verification of S follows immediately with no more resources needed. If one fails then one either suspends judgement on the truth value of S or concludes that it is false.

This argument also holds for proof procedures. The well known recursive enumerability and non recursive nature of proofs shows up in the enumeration carried out by a proof procedure and not in trying lots of procedures. This is based on the observation that the resources needed to verify or refute $Th_r(G(S))$, Eq. 10, are about[42] the same as are required to determine the truth value of $Y(P, P_{s-t}, G(S), x)$, Eq. 9, for the least resource intensive procedures. The quantification over space time locations of the implementation operation is taken care of by including in the axioms the statements of homogeneity and isotropy of space and time. It follows from this that the resources required to verify or refute a statement are independent of where and when the appropriate procedures are implemented.

The value of $r(S)$, Eq. 12, represents the least value of r for which the statement S appears in D_r . All D_r with $r \geq r(S)$ contain S , and S is not in any D_r where $r < r(S)$. In this sense $r(S)$ is the value of first appearance of S in the D_r . The same argument holds for theorems. If S is a theorem of T_r then $r(S)$ is the r value of first appearance of S as a theorem in T_r .

It is of interest to note that sentences S that are theorems have two r values of first appearance. The first value, which is usually quite small, is the smallest r value such that S , as a language expression, first appears in L_r . The second much larger value is the value at which S first becomes a theorem of T_r . If S is not a theorem, then the second value is the value at which S first appears in D_r .

In a similar vein, the elementary particles of physics have resource values of first appearance in the D_r . To see this let S be an existence statement for a particle type, such as a positron. Positrons exist only in those domains D_r such that $r \geq r(S)$. Statements regarding various properties of positrons also have r values of first appearance. All these values are larger than $r(S)$.

It should be noted that it is likely that there is no way to determine the values of $r(S)$ or r values of first appearance of various properties. Even if it were possible, one would have the additional problem of determining which procedure is most efficient.

E. Resource Limitations on Language Expressions

As was noted earlier the physical nature of language limits T_r in that all expressions as strings of alphabet symbols in L_r are limited to those requiring at most r resources to create, display, and manipulate the expressions. This includes all symbol strings, as outputs and as formulas or words in L_r .

To understand this better, for each a in the alphabet \mathcal{A} of L_r , let P_a be a procedure whose purpose is to create a physical system in some state that represents the symbol a . An expression X of length $n = L(X)$ of symbols in \mathcal{A} can be considered a function $X : \{1, 2, \dots, n\} \rightarrow \mathcal{A}$. Let p be an ordering rule for creating and reading X . For instance p can be a function from the natural numbers $1, 2, \dots$, to intervals of space and time where $p(1)$ is the space and time interval between $X(1)$ and $X(2)$ and \dots $p(n-1)$ is the interval between $X(n-1)$ and $X(n)$. As such p corresponds to a path along which the symbols of X are created and displayed. Let $P_{X,p}$ be the procedure whose purpose is to use the P_a to create X according to p .

The resources needed to implement $P_{X,p}$ depend on those needed to implement P_a and to construct X according to rule p . Let Δ be the amount of physical resources used for each implementation of P_a . Here $\Delta = \Delta_a$ is assumed to be independent of a . It includes the amount of space and other resources needed to display a symbol.

The amount of resources needed to create X is given by $L(I(P_{X,p}))\Delta + r'(P_{X,p})$. the first part is the resources used by the instruction string or program for $I(P_{X,p})$ and the second part includes the resources needed to carry out $I(P_{X,p})$ or do $P_{X,p}$ and follow path p . It does not include the resources needed to display X . These are given by $L(X)\Delta$.

As states of physical systems, symbols created in a noisy environment require energy resources to maintain. If a symbol requires δE energy resources per unit time interval to maintain, then maintaining an expression X for m time intervals requires a total of $mL(X)\delta E$ energy resources. This assumes that none of the energy is recoverable.

Putting the above together gives the result that the amount of resources needed to create, display, and maintain an expression X for m time intervals using instructions $I(P_{X,p})$ is given by

$$r_{X,m,P_{X,p}} = L(I(P_{X,p}))\Delta + r'(P_{X,p}) + L(X)\Delta + mL(X)\delta E. \quad (13)$$

This equation denotes a $2d+2$ dimensional equation with one for each $i = 1, 2, \dots, 2d+2$. Each component equation is given by

$$[r_{X,m,P_{X,p}}]_i = L(I(P_{X,p}))\Delta_i + [r'(P_{X,p})]_i + L(X)\Delta_i + mL(X)\delta E\delta_{i,2d+2}. \quad (14)$$

Here the subscripts i denote the i th component and $\delta_{i,2d+2} = 1(0)$ if $i = (\neq)2d+2$.

Any theory T_r with $r \geq r_{X,m,P_{X,p}}$ has $P_{X,p}$ in D_r . Also X is in L_r . Here and in the following, unless otherwise stated, relations between two values of r refer to all components of r . However, if $[r]_i < [r_{X,m,P_{X,p}}]_i$ for some i , then X is not in L_r as it requires too much of the i th component of resources to create, display, and maintain.

The previous discussion about minimal resources applies here in that there are many different procedures P' and instructions $I_{P'}$, for creating symbols, and many different reading rules, p' , and methods of maintaining X . The value of $r_{X,m,P'_{X,p'}}$ depends on all these parameters. Also different physical systems in different states, from very large to very small, can be used to represent the alphabet of L_r .

As before one is interested in the minimum value of $r_{X,m,P_{X,p}}$ for fixed X and m but varying P and p . Finding a minimum for the P and I_P variations may be hard as this includes the algorithmic complexity of X [29, 30, 31, 32]. However one would expect a minimal resource path p to be a geodesic. One also needs to account for variations in the extent and complexity of physical systems used to represent the alphabet symbols.

For very small symbols quantum effects become important. This is especially the case if symbols are represented by coherent states of quantum systems. Then the states must be protected against errors resulting from interactions with external fields and environmental systems. This is the basis for work on quantum error correcting procedures for quantum computers.

Here a fixed physical representation of alphabet symbols and a fixed path p are assumed. In this case Eq. 13 can be used to determine a number $N(r)$ that represents the maximum length of an expression X whose creation, display, and maintenance for a time r_{d+1} requires at most r resources. To this end one replaces $L(I(P_{X,p}))$ by its approximate upper limit $L(X)$. This accounts for the fact that, up to a constant, $L(I(P_{X,p}))$ is less than the length of a procedure that simply copies X . Also the X dependence of $r'(P_{X,p})$ is limited to a dependence on $L(X)$ only.

This allows one to define for each i a number N_i for any r by

$$N_i = \max_n [n\Delta_i + [r'(n,p)]_i + r_{d+1}n\delta E\delta_{i,2d+2} \leq r_i]. \quad (15)$$

N_i denotes the maximum length of any X such that the i th component of resources needed to create, display, and maintain X is $\leq r_i$. Also $L(X) = n$. $N(r)$ is defined by

$$N(r) = \min_{i=1,\dots,2d+2} N_i. \quad (16)$$

$N(r)$ is determined by the most resource intensive component to create, display, and maintain an expression relative to the available resources.

It should be noted that the resource limitations enter into L_r and T_r only through the requirement that the length $L(X)$ of all expressions in L_r is less than some $N = N(r)$. One also sees that for moderate and larger

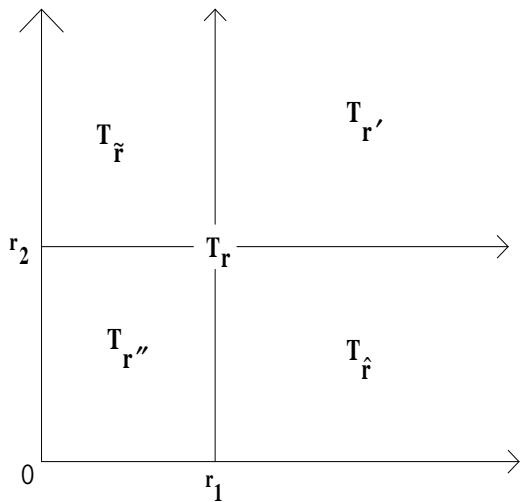


FIG. 2: Partial Ordering of the Theories on a two Dimensional Resource Space. Theories in the upper right quadrant, such as $T_{r'}$, are extensions or T_r . T_r is an extension of theories in the lower left quadrant such as $T_{r''}$. Theories in the other two quadrants are unrelated to T_r .

values of r , the value of $N = N(r)$ for most physical representations of language expressions is very large. As such it is a weak limitation especially when compared to the resources needed to determine the truth value of statements.

V. PARTIAL ORDERING OF THE T_r

The partial ordering of the resources $r = \{r_1, \dots, r_{2d+2}\}$ can be used to partially order the theories T_r . In particular it is assumed here that $T_r \supseteq T_{r'}$ if $r \geq r'$. Here $T_r \supseteq T_{r'}$ means that the domain of T_r includes that of $T_{r'}$ and that T_r is an extension of $T_{r'}$ in that every theorem of $T_{r'}$ is a theorem of T_r . The latter is based on the observation that the resource limitations are weaker for T_r than for $T_{r'}$. As a result every proof X of a theorem in $T_{r'}$ that does not include an axiom relating to resource limitations is a proof of the same theorem in T_r . Also axioms mentioning resource limitations have to be structured so that proofs including them do not generate contradictory theorems for different values of r . Whether this can be done or not is a problem for future work.

If r and r' are not in the domain of the partial ordering relation \geq , then the relation, if any, between T_r and $T_{r'}$ is undetermined. This would be the case, for example, if T_r has available twice the time resources and two thirds the space resources that are available to $T_{r'}$.

These relations are shown in Figure 2 where a two dimensional resource space is used to illustrate the relations. The figure coordinates show that the two resource

components are ≥ 0 . The lines drawn through T_r separate the theories into four quadrants. The theories in the upper right quadrant, denoted by $T_{r'}$, are all extensions of T_r , $T_r \subseteq T_{r'}$. T_r is an extension of all theories in the lower left quadrant, such as $T_{r''}$, or $T_{r''} \subseteq T_r$. The theories in the upper left and lower right quadrants, such as $T_{\hat{r}}$ and $T_{\hat{r}}$, are not related to T_r .

The locations of various theories of physics and mathematics in the partial ordering are determined by the resource limitations on the domains, theories, and languages. This includes limitations based on resource use to prove statements, to determine the truth value of statements, and to limit the length of language expressions.

One sees from this that expressions of a basic theory such as arithmetic are scattered throughout the T_r . There is no upper bound on the values of r below which all arithmetic expressions are found. It is also the case that for any r , no matter how large, almost all arithmetic expressions are found only in the $L_{r'}$ where $r' > r$. This holds even for the weak length limitation on expressions in the L_r . It is a consequence of the exponential dependence of the number of expressions on the expression length. The same holds for all names of the natural numbers as symbol strings in some basis.

Many expressions of theories based on the real and complex numbers, such as real and complex analysis, quantum mechanics, QED, and QCD are also scattered throughout the T_r . However these are limited to expressions that contain at most variables and names of special mathematical objects such as $e, \pi, \sqrt{2}, etc.$. These special objects are not random in that, for any n , they can be specified to n figures by an instruction set I_P as a symbol string of finite length that accepts n as input [29, 30, 31, 32]. Almost all of the mathematical objects, such as real numbers, complex numbers, functions, states, operators, etc., are random. Names for all of these cannot be found in any L_r no matter how large r is.

It follows that almost all sentences S in these theories are infinitely long. These expressions are in the limit language, L_∞ , only. They are not in L_r for any finite r .

Another way to state this is that quantum mechanics and many other other theories are limit theories. Each is a theory of first appearance for the parts of all the T_r that are expressions and theorems for the theory being considered. This holds even for arithmetic whose expressions, including names, are of finite but unbounded length.

VI. RESOURCE USE BY OBSERVERS

The resource space and the T_r , Figure 2, represent a background on which an intelligent system (or systems) moves in developing physical and mathematical theories and, hopefully, a coherent theory of physics and mathematics or a TOE. The main goal of interest for an observer (assumed equivalent to an intelligent system) or community of observers is to develop physical and math-

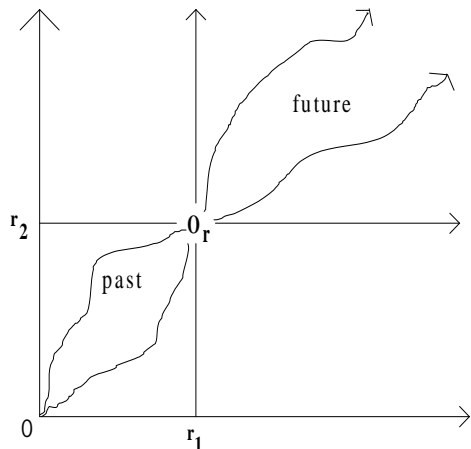


FIG. 3: Two Paths Showing Use of Physical Resources by an Observer. O_r shows the position of an observer after spending $r = p(t)$ resources by time t . Paths in the lower left and upper right quadrants, denoted as past and future, show use of resources at times before and after t . Also path gradients must be ≥ 0 everywhere.

ematical theories that explain their observations.

Here the need for observers to use physical resources to acquire this knowledge is emphasized. Observers start with elementary sense impressions and acts, uninterpreted by any theory, Sections II and III. They use physical resources to carry out experiments and theoretical calculations to develop physical theories that explain their impressions and suggest new ways to test the theories. Which resources an observer uses and what the resources are spent on are determined by the specific observer. It depends on choices made and the goal of the process for each observer.

It is clear from this that the process of using resources to develop a theory or theories to explain observations and results of experiments is a dynamical process. To this end let p denote a path in resource space that represents the resources used by an observer where $p(t)$ denotes the total amount of resources used up to time t by an observer. If $dp_i(t)/dt$ is the time rate of change of the use of the i th component of the resources then

$$p_i(t) = \int_0^t \frac{dp_i(t')}{dt'} dt' \quad (17)$$

gives the time development of the use of the i th resource component. The motion of an observer using resources can be shown on a figure similar to Fig. 2. This is done in Figure 3 which shows the location of an observer after having used $r = p(t)$ resources at some time t . As was done for Fig. 2, r is taken to be 2 dimensional. The figure shows two out of many possible paths in the resource space that an observer can follow. available for an observer. The path gradients, $dp_i(t)/dt$, are ≥ 0 everywhere. This follows from the requirement that used

resources cannot be recovered. Resources used before time t are in the lower left quadrant, labelled as past, and resources used after time t are in the upper right quadrant, labelled as future.

The knowledge gained by an observer O at time t after using $p(t)$ resources can be represented by a statement $\underline{S}_{p(t)} = \wedge_{j=1}^n S_j$ that is the conjunction of all statements verified or refuted by O after using $r = p(t)$ resources. That is

$$r(\underline{S}_{p(t)}) = p(t). \quad (18)$$

$\underline{S}_{p(t)}$ can include many types of component statements such as those about tests of agreement between theory and experiment. Also some or all of the component statements can be theorems. The number n of statements depends on many things including what procedures an observer decides to implement in acquiring knowledge.

As was noted before, Eq. 2, the amount of resources spent to verify or refute $\underline{S}_{p(t)}$ can be less than the sum of the resources needed to determine the truth value of each component S_j considered by itself. The amount of resources spent to verify or refute S_j in the conjunction is given here by $p(t_j) - p(t_{j-1})$ where t_j is the time resource used by O to verify or refute $\underline{S}_{p(t_j)} = \wedge_{k=1}^j S_j$.

The connection between O_r in Fig. 3 and T_r in Fig. 2 follows from Eq. 18. In particular $\underline{S}_{p(t)}$ is a statement in $D_{p(t)}$ and in $T_{p(t)}$. It is also the case that each component statement S_j in $\underline{S}_{p(t)}$ is in D_{Δ_j} and in T_{Δ_j} where $\Delta_j = p(t_j) - p(t_{j-1})$. Also any S_j that is a theorem is a theorem of T_{Δ_j} .

It follows from Eq. 2 that all component sentences of $\underline{S}_{p(t)}$ are included in D_r , L_r , and T_r where $r = p(t)$. T_r should prove some of the verified sentences in $\underline{S}_{p(t)}$ and prove none of the refuted sentences. Also T_r and D_r contain many other sentences obtained by observers following different resource use paths and choosing a different collection of statements to verify or refute. For instance if $\underline{S}'_{p'(t')} = \wedge_{j=1}^m S'_j$ is verified or refuted by following a different resource path p' , then $\underline{S}'_{p'(t')}$ and each component statement is in T_r and D_r provided that $p'(t') = r$.

VII. LOCAL REFLECTION PRINCIPLES

As is well known, the goal of any theory, including the T_r , is to determine the truth value of statements. The only method available for a theory to determine truth values is by proof of theorems. However this works if and only if the theory is consistent. All statements of inconsistent theories are theorems so there is no connection between theoremhood and truth or falseness.

This also applies to the partially ordered T_r . For this reason, it would be desirable if the T_r could prove their own consistency or validity. However, this is not possible for any theory, such as the T_r , containing some arithmetic [23, 24]. The same limitation applies also to any stronger theory that proves the consistency of the original theory.

It is assumed here that the resource limited T_r have the same properties regarding consistency proofs as theories with no resource limitations.

Here reflection principles, based on validity statements [26, 27], are used with the T_r to push validity proofs up in the partial ordering of the T_r . In this way theories higher up in the ordering can prove the validity of theories lower down. To this end let S be some statement such that T_r proves S , Eq. 7. Then $Th_r(G(S))$, given by Eq. 10 is a theorem of T_r . This is expressed by $T_r \vdash Th_r(G(S))$, which says that the sentence $Th_r(G(S))$ is a theorem of T_r , or that T_r proves that it proves S .

The validity of T_r at S is expressed by

$$Val_r(G(S)) \equiv (Th_r(G(S)) \implies S). \quad (19)$$

$Val_r(G(S))$ is a sentence in L_r which can be interpreted through G to say that if T_r proves S is a theorem, then S is true. Here one is using Tarski's notation that assertion of a statement S is equivalent to the truth of S [35]. This means that if $Val_r(G(S))$ were a theorem of T_r , then one could conclude from $T_r \vdash Th_r(G(S))$ that T_r proves the truth of S .

The problem is that because T_r cannot prove its own consistency it cannot prove validity statements such as $Val_r(G(S))$. Reflection principles [26, 27] are used here to extend the T_r with validity statements for the sentences in T_r . Because of resource limitations, the extensions must be considered separately for each S rather than adding validity statements for all sentences of T_r to the axioms of T_r [26, 27]. Also since the axiom sets Ax_r are not specified in any detail, the addition is taken care of by requiring that the axiom sets Ax_r are such that theories higher up in the partial ordering can prove the validity of theories lower down.

In this case the T_r have the property that for each S for which Eq. 10 holds, there exists a theory $T_{r'}$ with $r' > r$ that proves the validity of T_r at S or

$$T_{r'} \vdash Val_r(G(S)). \quad (20)$$

Since $r' > r$ implies that

$$T_{r'} \vdash Th_r(G(S)), \quad (21)$$

one has that $T_{r'} \vdash S$. In this way $T_{r'}$ reflects the validity of T_r and proves that S is true.[43]

This transfers the validity problem to $T_{r'}$. In order to conclude that S is true, one needs to prove that $T_{r'}$ is valid at $Th_r(G(S))$ and at $Val_r(G(S))$. This leads to an iterated application of the reflection principles generating a sequence of theories T_{r_n} where $r_{n+1} > r_n$ and $T_{r_{n+1}}$ proves the validity of the relevant statements for T_{r_n} . Based on Gödel's second incompleteness theorem [23, 24] the iteration process does not terminate. Here this leads to limit theories that have the same problem. The limit theories are the usual theories with no bounds on the available resources.[44]

VIII. POSSIBLE APPROACH TO A COHERENT THEORY OF PHYSICS AND MATHEMATICS

At this point little can be said about the details of a coherent theory of mathematics and physics or a TOE. However there are some properties of a TOE that would be expected if the partial ordering of theories and resource used by observers described here has merit. These are the relation of a coherent theory to the T_r and the problem of consistency.

A. Limit Aspects

As was seen in section V the expressions of arithmetic and other theories of physics and mathematics are scattered throughout the T_r with the number of expressions and sentences first appearing in T_r increasing exponentially with the value of r . This holds for arithmetic sentences and sentences of other theories with names of objects that are not random. However since names of most mathematical objects are infinitely long, so are sentences that include these names.

As was noted earlier, it follows from this that theories of physics and mathematics with no resource limitations are limit theories or theories of first appearance of all the expressions appropriate to the theory being considered. Arithmetic is the theory of first appearance of all the arithmetic expressions of the T_r . Quantum mechanics is the theory of first appearance of all expressions in the parts of the T_r that deal with quantum mechanics. The same holds for other theories. They are all limit theories or theories of first appearance of the relevant parts of the T_r .

If one follows this line of thought, then a coherent theory of mathematics and physics or TOE would also be a limit theory with expressions scattered throughout the partial ordering. In this case one would expect the TOE to be a common extension of all the T_r rather than of just parts of each T_r . In this case one expects that

$$T_r \subset TOE \quad (22)$$

holds for each r . That is any statement that is a theorem in some T_r is also a theorem in TOE. This requires careful inclusion of the resource limitations into the T_r and the Ax_r so that some obvious, and not so obvious, contradictory statements do not become theorems. Whether or not the TOE satisfies this condition has to await future work.

B. Consistency and a Coherent Theory

Consistency poses a problem for a coherent theory of physics and mathematics or a TOE to the extent that this theory is assumed to really be a final theory [10] in that it has no extensions. It was seen that Gödel's

incompleteness theorem on consistency [23, 24] and the use of reflection principles [26, 27] push the consistency problem up the network but never get rid of it. Also it follows directly from Eq. 22 (and from the fact that a TOE includes arithmetic) that a TOE cannot prove its own consistency.

This is problematic if a TOE is a final theory because if one extends a TOE to a theory proving that the TOE is consistent then a TOE is not a theory of everything. It is a theory of almost everything. And the same problem holds for the extension.

This situation is unsatisfactory. However it is no worse than the existing situation regarding other theories such as arithmetic, quantum mechanics, and many other physical and mathematical theories. Each of these theories can express their own consistency, so none of them can prove their own consistency [23, 24]. Such proofs must come from stronger theories which then have the same problem. Of course, there is no reason to doubt the consistency of these theories, and their usefulness shows that they are almost certainly consistent.

For a limit or final theory [10] one would like to do better and not leave the problem hanging. One solution might be to solve the problem axiomatically by including an axiom that asserts the existence of a consistent coherent theory of physics and mathematics. How the axiom is stated, such as whether or not it is in essence the strong anthropic principle [8, 9, 25], and the usefulness of this approach, will be left to future work.

IX. SUMMARY AND FUTURE WORK

A. Summary

A partial ordering of resource limited theories and their extensions has been studied as a possible approach to a coherent theory of physics and mathematics. Each theory T_r , domain D_r , and language L_r has a limited amount r of space, time, momentum, and energy resources available.

The resource limitations on the D_r restrict all statements S in D_r to require at most r resources to verify or refute. The statements can refer to processes, physical systems, purposes of processes, implementations of procedures, and outcomes of experiments and whether they agree or disagree with theoretical predictions.

Resource limitations on the T_r require that all theorems are provable using at most r resources. Also if T_r is consistent, then all theorems of T_r must be true in D_r .

A less restrictive limitation is that the language L_r is limited to expressions, as strings of symbols from some alphabet, that require less than r resources to create, display, and maintain. This is expressed here by a length limitation on the expressions, given by Eq. 16, that is based on the essential physical nature of language [20].

The contents of the theories are described in some detail. Included are procedures, equipment, instructions

for procedures and purposes. The implementation operation and its role in the use of resources is discussed. These components were used to give statements in L_r that express agreement between theory and experiment, and provability of a statement S . The role of Gödel maps based on the physical nature of language in the provability statement was noted.

It was noted that there are many different procedures for determining the truth value of a statement S . As a result there is a minimum amount $r(S)$ of physical resources associated with determining the truth value of S . Based on this $r(S)$ is also the resource value of first appearance of S in the D_r and T_r . If S refers to the existence of some elementary particle of physics then the particle first appears in $T_{r(S)}$ and in $D_{r(S)}$.

A partial ordering of the theories is based on the partial ordering of the resources r . $T_{r'}$ is an extension of T_r (all theorems of T_r are theorems of $T_{r'}$) if $r' \geq r$, i.e., if for all components r_i of r , $r'_i \geq r_i$. This requirement is a nontrivial condition that the axioms Ax_r of each T_r must satisfy. This is in addition to the requirement that no statement requiring $> r$ resources to verify or refute can be a theorem of T_r . Also no false statement in D_r can be a theorem of T_r .

The motion of an observer using resources to develop theories was briefly discussed. It was noted that the amount r of resources used by an observer can be divided into parts with each part being the resources used to verify or refute a statement. The collection of all statements verified or refuted by an observer, following some path p of resource use, represents the total knowledge of the observer regarding development of physical and mathematical theories.

A brief discussion was given of the use of reflection principles [26, 27] to push the effect of Gödel's second incompleteness theorem [23, 24] on the T_r up in the partial ordering. This was done by the use of validity statements $Val_r(G(S) \equiv Th_r(G(S)) \implies S$ which state that T_r is valid for S . Here it is assumed that the axioms Ax_r are such that for each S there is an $r' > r$ such that both $Val_{r'}(G(S))$ and $Th_{r'}(G(S))$ are theorems of $T_{r'}$. Gödel's theorem, applied to $T_{r'}$ leads to iteration of this process to limit theories with no bounds on the available resources.

The possible use of the partial ordering of the T_r as an approach to a coherent theory of physics and mathematics, or TOE, was briefly discussed. It was noted that a TOE must be a limit theory that includes all the T_r , i.e. $T_r \subset TOE$. In this way a TOE includes arithmetic, quantum mechanics and other physical and mathematical theories, which are also parts of the T_r . This introduces a problem for consistency. Since a TOE can express its own consistency, it cannot prove its own consistency. However if a TOE is a final theory with no extension, then the consistency problem for a TOE is left hanging.

B. Future Work

As the above suggests there is much to do. Probably the most important need is to extend the theories to include probability and information theory concepts. It is expected that this will be important relative to observers spending resources to acquire knowledge and move towards a limit theory.

Another basic need is to develop the description of the theories T_r so that they describe the use of resources and the effects of limited availability of resources. This is clearly necessary if the axioms of T_r are to be such that no statement requiring more than r resources to verify or refute is a theorem of T_r .

The conditions imposed on the axioms Ax_r in this work

are quite complex. At this point it is open if there even exist axiom sets that can satisfy all the conditions. This needs to be investigated.

Another assumption that must be removed is embodied in the use of Eq. 16 to limit the length of language expressions. The theories T_r must take account of the observation that physical representations of language symbols and expressions as symbol strings can vary widely in size and resource requirements to create, display, maintain, and manipulate. There is no physical principle preventing symbol sizes ranging from nanometers or smaller to kilometers or larger. It is possible that removal of this and the other assumptions may require much more development of the ideas presented here.

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- [37] It is tempting to combine momentum and energy with space and time and let r be a $d + 1$ dimensional vector $r_1, r_2 \dots, r_{d+1}$ where each r_j denotes the available number of phase space cells for the j th dimension, and d is the number of space dimensions. The number of phase space cells of unit volume \hbar^{d+1} associated with r is given by $N_r = \prod_{j=1}^d r_j$. Here \hbar is Planck's constant divided by 2π . However this will not be done here.
- [38] Quantum mechanical examples of language symbols and expressions include lattices of potential wells containing ink molecules and products of spin projection eigenstates of spin systems also localized on a lattice. More details are given in [25] and especially in [20].
- [39] Note that the instructions either have to specify the ordering of reading the output symbols or a standard ordering must be assumed. This is needed to convert the outcome $\{0, 0, 1, 1, 1, \}$, as an unordered collection of symbols, to the symbol string 01011.
- [40] Of course mathematical analysis deals easily with single symbol representations of real numbers such as $\pi, e, \sqrt{2}$ and their properties. But these are not outputs of measurements or equipment readings.
- [41] For astronomical systems, state preparation is not possible.
- [42] This allows for the small amount of additional resources needed to prove the quantified statement.
- [43] One cannot conclude directly from Eq. 7 that S is true because T_r lacks a proof of its own validity at S .
- [44] Iteration of this process into the transfinite by use of constructive ordinals [26, 36] and closure by the use of self truth axioms is discussed in the literature [27].